Chapter 6
Frequency Response, Bode Plots and Resonance

*Signal Processing* is concerned with manipulating Signals to extract *Information* and to use that information to generate other useful Signals.
Goal

1. Fundamental Concepts of Fourier Analysis.

2. Filter of a given input consisting of sinusoidal components using the filter’s transfer function.

3. Low-pass or High-pass filter & Transfer Functions.


5. Understand Various Filters

6. Simple Digital Signal Processing System
Fourier Analysis

All Real-World Signals are Linear Sums of Sinusoidal components having various frequencies, amplitudes & phases.

\[ f(t) = \frac{a_o}{2} + \sum_{n} [a_n \cos n\omega t + b_n \sin n\omega t] \]

\[ a_n = \frac{2}{\tau} \int_{t_o}^{t_o+\tau} f(t) \cos n\omega t \, dt \quad b_n = \frac{2}{\tau} \int_{t_o}^{t_o+\tau} f(t) \sin n\omega t \, dt \quad \omega = \frac{2\pi}{\tau} \]

(a) Music waveform
(b) Sinusoidal components
Even Step function can be analyzed

(a) Periodic square wave

(b) Several of the sinusoidal components and the sum of the first five components
Let $\omega = 2\pi / \tau$

$$f(t) = \frac{a_0}{2} + \sum_{n} a_n \cos n\omega t + b_n \sin n\omega t$$

$$a_n = \frac{2}{\tau} \int_{t_0}^{t_0+\tau} dt \ f(t) \cos n\omega t$$

$$b_n = \frac{2}{\tau} \int_{t_0}^{t_0+\tau} dt \ f(t) \sin n\omega t$$

$$F(\omega) \text{ or } f(t)$$

---

**Fourier Analysis of Music**

*Time t*  

$a_n, b_n$

*A kind of Spectrum Analyzer*
<table>
<thead>
<tr>
<th>Signal</th>
<th>Frequency Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electrocardiogram</td>
<td>0.05 to 100 Hz</td>
</tr>
<tr>
<td>Audible sounds</td>
<td>20 Hz to 15 kHz</td>
</tr>
<tr>
<td>AM radio broadcasting</td>
<td>540 to 1600 kHz</td>
</tr>
<tr>
<td>Video signals (U.S. standards)</td>
<td>Dc to 4.2 MHz</td>
</tr>
<tr>
<td>Channel 6 television</td>
<td>82 to 88 MHz</td>
</tr>
<tr>
<td>FM radio broadcasting</td>
<td>88 to 108 MHz</td>
</tr>
<tr>
<td>Cellular radio</td>
<td>824 to 891.5 MHz</td>
</tr>
<tr>
<td>Satellite television downlinks (C-band)</td>
<td>3.7 to 4.2 GHz</td>
</tr>
<tr>
<td>Digital satellite television</td>
<td>12.2 to 12.7 GHz</td>
</tr>
</tbody>
</table>
Filters

Goal: Retain & Reject the Components in Certain Frequency ranges

Filters process the Sinusoid Components of an input signal differently depending on the frequency of each component.

"Code" as Political Filter!
Filters Types

(a) Lowpass

(b) Highpass

(c) Bandpass

(d) Band reject

Example of Ideal Low Pass filter
Transfer Functions

Transfer function $H(f)$ of Two-Port Filter:
\[ H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} \]

$|H(f)|$: Amplitude Change of Each Frequency Component

$\angle H(f)$: Phase Change of Each Frequency Component

---

Graphs showing $|H(f)|$, $H(f)$, and $\angle H(f)$ as functions of frequency $f$.
Filter of Multiple Components

1. Determine frequency & Phasors for each input component.

2. Determine Complex Transfer function for each component.

3. Obtain the phasors for each output component by multiplying the phasors for each input component by transfer-function.

4. Convert the phasors for each output components into time functions. Add these time functions to produce the output.
Experimental Determination of $H(f)$

- Application of Sinusoidal Source at fixed frequency
- Measurement of Output Signal Amplitude, Phase
- Procedure is repeated for each frequency of interest
Active Noise Canceller

Noise Reduction in Airplane or Car
- Passive : Absorber
- Active : Canceller

Measurement of Signal
- Source : Engine
- Receiver : Seat
  Best Cancellation to Passenger

These sound waves partially cancel those from the engine
Low Pass Filter
First-Order Low Pass Filters

Total Impedance: \( Z_t = Z_R + Z_C = R + \frac{1}{j\omega C} = R + \frac{1}{j2\pi fC} \)

\[
I = \frac{V_{in}}{Z_t} = \frac{V_{in}}{R + \frac{1}{j2\pi fC}}
\]

\[
V_{out} = IZ_C = \frac{1}{j2\pi fC} I
\]

\[
V_{out} = \frac{1}{j2\pi fC} \times \frac{V_{in}}{R + \frac{1}{j2\pi fC}}
\]

\[
H(f) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j2\pi fRC}
\]
First-Order Low Pass Filters

\[
H(f) = \frac{1}{1 + j(f/f_B)} \quad f_B = \frac{1}{2\pi RC}
\]

\[
|H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}}
\]

\[
\angle H(f) = -\arctan\left(\frac{f}{f_B}\right)
\]
Low Pass RC & RL Filters

\[ H(f) = \frac{1}{1 + j\left(\frac{f}{f_B}\right)} \]

\[ f_B = \frac{1}{2\pi RC} \]

\[ f_B = \frac{R}{2\pi L} \]
Decibels

\[ |H(f)|_{\text{dB}} = 20 \log |H(f)| \]

| \( |H(f)| \) | \( |H(f)|_{\text{dB}} \) |
|---|---|
| 100 | 40 |
| 10  | 20 |
| 2   | 6  |
| \( \sqrt{2} \) | 3 |
| 1   | 0  |
| 1/\( \sqrt{2} \) | -3 |
| 1/2 | -6 |
| 0.1 | -20 |
| 0.01| -40 |

![Graph showing linear and decibel scale](image)

Clear Description of Frequency of Certain Range of frequency
Cascaded Two-Port Networks

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{V_{out}}{V_{in1}} = \frac{V_{out1}}{V_{in1}} \times \frac{V_{out2}}{V_{in2}} \]

\[ H(f) = H_1(f) \times H_2(f) \]

\[ |H(f)|_{dB} = |H_1(f)|_{dB} + |H_2(f)|_{dB} \]
Logarithmic Frequency Scales

On a logarithmic scale, the variable is multiplied by a given factor for equal increments of length along the axis.

Decade: a range of frequencies for which the ratio of the highest frequency to the lowest is 10.

Number of Decades = \[ \log \left( \frac{f_2}{f_1} \right) \]

Octave: a two-to-one change in frequency for clear description of low range frequency such as 12 to 20 MHz.

Number of Octaves = \[ \log_2 \left( \frac{f_2}{f_1} \right) = \left( \frac{\log(f_2/f_1)}{\log(2)} \right) \]
**Bode Plots**

Magnitude in Decibels versus a logarithmic scale frequency.

\[
|H(f)|_{dB} = 20 \log |H(f)| \quad \text{vs} \quad \log f
\]

In Low-Pass filter

\[
H(f) = \frac{1}{1 + j(f / f_B)} \quad \Rightarrow \quad |H(f)|_{dB} = -10 \log \left[ 1 + \left( \frac{f}{f_B} \right)^2 \right]
\]

\[
|H(f)|_{dB} \approx 0 \quad \text{at} \quad f = f_B
\]

\[
|H(f)|_{dB} = -3 \text{dB at} \quad f = f_B
\]

\[
f_B = \text{Corner frequency} \quad \text{Breaking Frequency}
\]

\[
|H(f)|_{dB} = -20 \log \left( \frac{f}{f_B} \right)
\]
Phase Bode Plot

1. A horizontal line at zero for \( f < \frac{f_B}{10} \).
2. A sloping line from zero phase at \( f_B/10 \) to \(-90^\circ\) at \( 10f_B \).
3. A horizontal line at \(-90^\circ\) for \( f > 10f_B \).
Example of Bode Plot

\[ R = \frac{1000}{2\pi} = 159 \ \Omega \]

\[ \begin{align*}
V_{\text{in}} & \quad + \\
\quad - \\
\hspace{1cm} V_{\text{out}} \\
\end{align*} \]

\[ C \]

\[ 1 \ \mu F \]

\[ 20 \log \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| \]

\[ \begin{align*}
0 & \quad 100 & \quad 1000 & \quad 10^4 & \quad 10^5 \\
\end{align*} \]

\[ f \]

\[ \begin{align*}
0 & \quad 100 & \quad 1000 & \quad 10^4 & \quad 10^5 \\
0^\circ & \quad -45^\circ & \quad -90^\circ \\
\end{align*} \]

Phase
First-Order High-Pass Filters

Total Impedance: \( Z_t \)

\[
Z_t = Z_R + Z_C = R + \frac{1}{j\omega C} = R + \frac{1}{j2\pi fC}
\]

\[
I = \frac{V_{in}}{Z_t} = \frac{V_{in}}{R + \frac{1}{j2\pi fC}}
\]

\[
V_{out} = R \times \frac{V_{in}}{R + \frac{1}{j2\pi fC}}
\]

\[
H(f) = \frac{V_{out}}{V_{in}} = \frac{j2\pi fRC}{1 + j2\pi fRC}
\]
High Pass RC Filters

\[ H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{j(f/f_B)}{1 + j(f/f_B)} \quad f_B = \frac{1}{2\pi RC} \]

\[ |H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \]

\[ \angle H(f) = 90° - \arctan\left(\frac{f}{f_B}\right) \]
High Pass RC & RL Filters

\[ V_{\text{out}} f_B = \frac{1}{2\pi RC} \]

\[ R C \]

\[ j \frac{1}{2\pi fC} \]

\[ V_{\text{in}} \]

\[ V_{\text{out}} \]

\[ f_B = \frac{R}{2\pi L} \]

\[ j2\pi fL \]

\[ f \]

\[ \frac{f}{f_B} \]
Bode Plot of High Pass filter

(a) Magnitude

(b) Phase

- High-frequency asymptote
- Low-frequency asymptote
- Slope = 20 dB/decade
- 3 dB
- Straight-line approximation
- Actual phase

\( |H(f)|_{\text{dB}} \)

\( \angle H(f) \)

\( f_B/100 \) \quad \( f_B/10 \) \quad \( f_B \) \quad \( 10f_B \) \quad \( 100f_B \)
Summary

High Pass Filter

\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = \frac{f / f_c}{\sqrt{1 + \left( f / f_c \right)^2}}
\]

Low Pass Filter

\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = \frac{1}{\sqrt{1 + \left( f / f_c \right)^2}}
\]

Red Line: Multiplication

\[
\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right| = \frac{2(f / f_c)}{\sqrt{1 + \left( f / f_c \right)^2}}
\]

Red Line shows Band Selection!

\[ f_c = 1/(2\pi RC) \]
Band Pass RC Filters with Cascade Network

\[ \frac{|V_{out}|}{V_{in}} = \frac{f_H f}{\sqrt{(f^2 - f_H f_L)^2 + [f_L + (1+r)f_H]^2}} f^2 \]

\[ f_H = \frac{1}{2\pi R_H C_H} \quad f_L = \frac{1}{2\pi R_L C_L} \quad r = \frac{R_H}{R_L} \]

\( f_L \leq f \leq f_H : \text{Pass Band} \)

Response of a band-pass filter

[Diagram of Band Pass RC Filters with Cascade Network]
Band Reject RC Filters

\[ f_L = \frac{1}{2\pi R_H C_H} \]

\[ f_H = \frac{1}{2\pi R_L C_L} \]

Not effective combination→Use Active filter!
RC Circuit as Differentiator

Voltage across C: \( V_{in} - V_{out} \)

Voltage across R: \( V_{out} \)

\[
I = C \frac{d}{dt} [V_{in} - V_{out}] = \frac{V_{out}}{R}
\]

For small RC

\[
\frac{V_{out}}{RC} = \frac{d}{dt} [V_{in} - V_{out}] \rightarrow \frac{dV_{in}}{dt} \gg \frac{dV_{out}}{dt} \rightarrow C \frac{dV_{in}}{dt} = \frac{V_{out}}{R}
\]

\[
V_{out} = RC \frac{dV_{in}}{dt}
\]

Differentiation of Input Voltage
Differentiation of Square Wave

Leading Edge Detector

\[ V_{in} \]

\[ V_{out} \]

\[ A \rightarrow 100\text{pF} \rightarrow B \rightarrow 10k \rightarrow C \]

- \( A \) - input
- \( B \) - \( RC \) time constant = 1\( \mu \)s
- \( C \) - output
RC Circuit as Integrator

Voltage across $R$ : $V_{in} - V_{out}$
Voltage across $C$ : $V_{out}$

$I = C \frac{dV_{out}}{dt} = \frac{V_{in} - V_{out}}{R}$

For Large RC

\[
\frac{V_{out}}{RC} = \frac{d}{dt}[V_{in} - V_{out}] \rightarrow V_{in} \gg V_{out} \rightarrow C \frac{dV_{out}}{dt} = \frac{V_{in}}{R}
\]

\[
V_{out} = \frac{1}{RC} \int V_{in} \, dt + \text{const}
\]
Integration of Input Voltage
Mechanical Resonance    Electrical Resonance

Wind & Bridge

Electron & RLC
Series Resonance

Series Resonance Frequency

\[ Z_s(f) = j2\pi f L + R - j\frac{1}{2\pi f C} \]

Resonance Frequency

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

Quality factor

\[ Q_s = \frac{2\pi f_0 L}{R} = \frac{1}{2\pi f_0 CR} \]

Total Impedance

\[ Z_s(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]
Impedance Characteristics

(a) Magnitude

(b) Phase
Filter Case of Series Resonance

- **Low Pass**
- **High Pass**
- **Band Pass**
- **Band Notch**

Diagram:

- \( V_{in} \)
- \( R \)
- \( L \)
- \( C \)
Second-Order Low Pass Filter

\[ Z_s(f) = j2\pi fL + R - j \frac{1}{2\pi fC} \]

\[ Z_s(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

\[ Q_s = \frac{2\pi f_0 L}{R} = \frac{1}{2\pi f_0 CR} \]

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{-jQ_s \left( \frac{f_0}{f} \right)}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]
First & Second Order Low Pass Filters

(a) Second-order lowpass filter

(b) First-order lowpass filter

(c) Transfer-function magnitudes

Q = 1
Q = 2
Q = 5

Slope = $-20$ dB/decade
Slope = $-40$ dB/decade
Second-Order High Pass Filter

\[ Z_s(f) = j2\pi f L + R - j \frac{1}{2\pi f C} \]

\[ Z_s(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

\[ Q_s = \frac{2\pi f_0 L}{R} = \frac{1}{2\pi f_0 CR} \quad f_0 = \frac{1}{2\pi \sqrt{LC}} \]

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{jQ_s \left( \frac{f}{f_0} \right)}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]

Inductor Component
Bode Plot of Second-Order High Pass Filter

\[ |H(f)| \text{ (dB)} \]

频率响应曲线展示了不同品质因子 \( Q_s \) 的高通滤波器的Bode图。图中显示了以下品质因子对应的响应曲线：

- \( Q_s = 5 \)
- \( Q_s = 2 \)
- \( Q_s = 1 \)
- \( Q_s = 0.5 \)

频率 \( f \) 范围从 \( f_0/10 \) 到 \( 100f_0 \)。
Series Resonant Circuit as a Band-Pass Filter

\[ Z_s(f) = j2\pi f L + R - j \frac{1}{2\pi f C} \]

\[ Z_s(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

Resistor Component

\[ Q_s = \frac{2\pi f_0 L}{R} = \frac{1}{2\pi f_0 C R} \]

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

\[ \frac{V_{out}}{V_{in}} = \frac{1}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]
Bode Plot of Second-Order Band Pass Filter

\[ B = f_H - f_L = \frac{f_0}{Q_s} \]

When \( Q_s \gg 1 \),

\[ f_H \approx f_0 + \frac{B}{2} \]
\[ f_L \approx f_0 - \frac{B}{2} \]
Series Resonant Circuit as Band-Reject Filter

\[ Z_s(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

\[ Z_{LC} = Z_L + Z_C = j[2\pi f L - \frac{1}{2\pi f C}] \]

\[ Q_s = \frac{2\pi f_0 L}{R} = \frac{1}{2\pi f_0 CR} \quad f_0 = \frac{1}{2\pi \sqrt{LC}} \]

\[ \frac{V_{out}}{V_{in}} = \frac{jQ_s \left( f / f_0 - f_0 / f \right)}{1 + jQ_s \left( f / f_0 - f_0 / f \right)} \]
Bode Plot of Second-Order Band Reject Filter

$|H(f)|$ (dB)

$Q_s = 5$, $Q_s = 2$, $Q_s = 1$, $Q_s = 0.5$

$f_0/10$, $f_0$, $10f_0$
**PARALLEL RESONANCE**

\[ Z_p = \frac{1}{(1/R) + j2\pi fC - j(1/2\pi fL)} \]

- **Quality factor** \( Q_p = \frac{R}{2\pi f_0 L} = 2\pi f_0 CR \)
- **Resonance frequency** \( f_0 = \frac{1}{2\pi \sqrt{LC}} \)
- **Total Impedance** \( Z_p = \frac{R}{1 + jQ_p \left( f/f_0 - f_0/f \right)} \)
Parallel Resonant Circuit as Band-Pass Filter

\[ V_{out} = \frac{IR}{1 + jQ_p \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]

\[ B = f_H - f_L = \frac{f_0}{Q_p} \]
Resonance Filter

\[ Z_{\text{total}} = Z_R + Z_{LC} \]
\[ \frac{1}{Z_{LC}} = \frac{1}{Z_L} + \frac{1}{Z_C} = \frac{1}{j2\pi fL} - \frac{2\pi fC}{j} = j[2\pi fC - \frac{1}{2\pi fL}] \]
\[ Z_{\text{total}} = R + \frac{j}{(1/2\pi fL) - 2\pi fC} \]

\[ Q = 2\pi f_o RC \quad f_o = 1/2\pi \sqrt{LC} \]

\[ \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_{LC}}{Z_R + Z_{LC}} = \frac{1}{1 - jQ_s (f/f_o - f_o/f)} \]
SAW Filter

SAW: Surface Acoustic Wave

Input IDT

Output IDT

IDT: Interdigital Transducer

V₀: SAW Velocity of free surface

λ: Wavelength (= V₀/f₀)

f₀: Center frequency

Consumer Applications
- TV/VCR
- CATV Converter
- Satellite TV
- Wireless Remote Control
- Cordless Phone
- Pager
- Cellular Phone
- PCS
- WLL
- IMT-2000

Commercial Applications
- LAN
- Fiber Optic Repeater
- TV Test Equipment
- Clock Recovery Filter
- Spectrum Analyzer

Military Applications
- Navigation
- Weather Balloon
- Radar
- Chirp Filter
- Filter Bank
- Convolver
- Delay Line
If a signal contains no components with frequencies higher than $f_H$, the signal can be exactly reconstructed from its samples, provided that the sampling rate $f_s$ is selected to be more than twice $f_H$. 

$f_s > 2 f_H$ : Sampling Rate (frequency)
Example of ADC (3bit ADC)

Sampling Error

\[ T = \frac{1}{f_s} \]

Quantization Error

\[ N = 2^k \]
Digital Low Pass Filter

\[
\begin{align*}
R \\
C \\
y(t)
\end{align*}
\]

\[y(t) - x(t) + \tau \frac{dy(t)}{dt} = 0\]

KCL at Top Node

\[
\frac{y(t) - x(t)}{R} + C \frac{dy(t)}{dt} = 0
\]

\[
y(t) - x(t) + \tau \frac{dy(t)}{dt} = 0
\]

with \(\tau = RC\)

\[
\frac{dy(t)}{dt} \approx \frac{\delta y(t)}{\delta t} = \frac{y(n) - y(n-1)}{T}
\]

\[
y(n) = ay(n-1) + (1 - a)x(n)
\]

\[
a = \frac{\tau/T}{1 + \tau/T}
\]
Application of Digital Low Pass filter to Step Signal

Step function must be smoothed by Low Pass filter, leading to Constant value

\[ y(n) = ay(n-1) + (1-a)x(n) \]
\[ a = \frac{\tau/T}{1 + \tau/T} \]
Signal with Typical Noise

(a) Signal

(b) 60-Hz interference

(c) Acoustic noise

(d) Simulated pressure-sensor output $x(t)$
Filtered Signal

\[ x(n) \xrightarrow{\text{60-Hz notch filter}} z(n) \xrightarrow{\text{Lowpass filter}} y(n) \]

Figure 6.49: Output signal.